VECTOR FIELDS ON MAPPING SPACES AND A CONVERSE TO THE AKSZ CONSTRUCTION

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Abstract. The well-known AKSZ construction (for Alexandrov–Kontsevich–Schwarz–Zaboronsky) gives an odd symplectic structure on a space of maps together with a functional S that is automatically a solution for the classical master equation $(S,S)=0$. The input data required for the AKSZ construction consist of a volume element on the source space and a symplectic structure of suitable parity on the target space, both invariant under given homological vector fields on the source and target. In this note, we show that the AKSZ setup and their main construction can be naturally recovered from the single requirement that the 'difference' vector field arising on the mapping space be gradient (or Hamiltonian). This can be seen as a converse statement for that of AKSZ. We include a discussion of properties of vector fields on mapping spaces.

1. Introduction

1.1. **The AKSZ construction.** What is now known as the 'AKSZ construction' was introduced by Alexandrov, Kontsevich, Schwarz and Zaboronsky in [1], who applied it to some models of topological field theory. The AKSZ construction was further elaborated, for example, in [3] and [8]. It is a construction of a particular solution of the classical master equation on a space of fields together with the equation itself (i.e., an odd symplectic structure) from certain data on the source and target supermanifolds.

More precisely, the AKSZ construction starts from two supermanifolds, M and N (the source and target, respectively), together with the following **input** data:

- a homological vector field Q_1 on M ,
- a homological vector field Q_2 on N ,
- a volume element $\boldsymbol{\rho}=\rho(x)Dx$ on M ,
- a symplectic 2-form ω of parity $q+1$ on N ,

where $\dim M = p|q$, and such that $\boldsymbol{\rho}$ is invariant under Q_1 and ω is invariant under Q_2. It follows that the vector field Q_2 is locally Hamiltonian w.r.t. the symplectic structure ω, so that locally $i_{Q_2}\omega=-dH$ for some H, where $\tilde{H}=q$. (By the tilde we denote the parities of the objects in question.)

Then as an **output** the following objects on the space of maps $\mathbf{Map}(M,N)$ are obtained: the 2-form

$$\Omega[\varphi,\delta\varphi]=\int_M Dx\,\rho(x)\,\omega\big(\varphi(x),\delta\varphi(x)\big) \tag{1}$$

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and the functional

$$S[\varphi] = \int\limits_M Dx\, \rho(x)\, \Bigl(Q_1^a(x)\,\frac{\partial\varphi^i}{\partial x^a}\lambda_i(\varphi(x)) - H(\varphi(x))\Bigr)\,, \tag{2}$$

where $\omega = d\lambda$ (locally), so that the form Ω defines an odd symplectic structure on $\mathbf{Map}(M,N)$ and the functional S satisfies the classical master equation

$$(S,S) = 0 \tag{3}$$

w.r.t. the corresponding odd Poisson bracket.

Here the mapping space $\mathbf{Map}(M,N)$ is considered as an infinite-dimensional supermanifold. Functions $\varphi^i(x)$, some of which may be odd, defining a map $\varphi\colon\, M \to N$ in local coordinate systems on M and N, are regarded as 'coordinates' on $\mathbf{Map}(M,N)$. A form on $\mathbf{Map}(M,N)$ is by definition a function on the antitangent bundle $\Pi T\,\mathbf{Map}(M,N)$. The functions $\varphi^i(x)$ and their variations $\delta\varphi^i(x)$, to which we ascribe parities opposite to those of $\varphi^i(x)$, are together 'coordinates' on the infinite-dimensional supermanifold $\Pi T\,\mathbf{Map}(M,N)$. This usage agrees with the standard language in field theory and integrable systems. We use Dx (with the capital D) as the notation for the Berezin volume element.[1]

Note that the functional (2), known as the AKSZ action, is multi-valued, but its variation is well-defined. The study of such multi-valued functionals was initiated by Novikov in early 1980s, see, e.g., [6]. Formulas such as (1) for a symplectic structure are known. To them correspond the so-called "ultra-local" field-theoretic Poisson brackets, see [4].

The AKSZ construction has found numerous applications. In the original paper [1], the authors applied it to the Chern-Simons model of topological quantum field theory and to some other models. It was applied to deformation quantization by Cattaneo and Felder [3].

1.2. **Our claim and the structure of the paper.** In this note, we show that the AKSZ construction follows naturally from a very simple setup.

Recall that, given two vector fields X and Y on (super)manifolds M and N, there is a construction of an induced vector field on the mapping space $\mathbf{Map}(M,N)$. We denote it by $d(X,Y)$ and call the *difference construction* for X and Y. At each point $\varphi \in \mathbf{Map}(M,N)$, the value of $d(X,Y)$ is defined as the difference $Y\circ\varphi - d\varphi\circ X$, which measures the failure of X and Y to be φ-connected.[2] Suppose we take homological vector fields $Q_1 \in \mathrm{Vect}(M)$ and $Q_2 \in \mathrm{Vect}(N)$. We claim that the single requirement that the vector field $d(Q_1,Q_2)$ on $\mathbf{Map}(M,N)$ be "gradient" or "Hamiltonian", i.e., come from the variation of some action, allows to recover the whole AKSZ setting, including formulas (1) and (2) for the symplectic form Ω and the master action S.

In more detail, this goes as follows.

The master equation $(S,S)=0$ for an action S is equivalent to the corresponding Hamiltonian vector field X_S being homological.

[1] We avoid the notation 'dx' for a volume element because of the contradictions with the transformation law under a change of coordinates. The capital letter in Dx should not be confused with the notation for a path integral measure element, e.g., in $\mathcal{D}\varphi$

[2] Considering $Y\circ\varphi - d\varphi\circ X$ as a tangent vector to the space of maps at φ is undoubtedly classical, but treating it as a vector field with variable φ, which is a certain shift of view, probably belongs to [1].

In [1], it was shown that the vector field X_S corresponding to the AKSZ action (2) is precisely the difference construction $d(Q_1, Q_2)$ for the homological vector fields Q_1 and Q_2. One can see that the difference construction for homological fields is automatically homological. Therefore the AKSZ action satisfies the master equation. This was the argument in [1].

In the present paper, we show that, conversely, one may start from just two homological vector fields Q_1 and Q_2 (without initially assuming any other ingredients of the AKSZ scheme) and require simply that the vector field $d(Q_1, Q_2)$ on $\mathbf{Map}(M, N)$, which is automatically homological, can be written in the gradient form

$$d(Q_1, Q_2) = \int_M Dx \Psi^{ij}(x, \varphi(x)) \frac{\delta S}{\delta\varphi^j(x)} \frac{\delta}{\delta\varphi^i(x)},$$

for some functional $S[\varphi]$, where no a priori properties such as symmetry or Jacobi identity are assumed for the object $\Psi^{ij}(x, y)$. Then it turns out that $\Psi^{ij}(x, y)$ automatically comes from an odd symplectic structure on $\mathbf{Map}(M, N)$ and we recover uniquely all the AKSZ formulas (in a slightly generalized form).

The structure of the paper is as follows.

In Section 2 we review some general facts about vector fields on spaces of maps. Most of them are known, but we felt that it would be useful to have them assembled together.

In Section 3 we explain our main construction.

Throughout this note, we follow notations and conventions concerning supermanifolds, homological vector fields and even or odd Poisson brackets that can be found, e.g., in [9] and [11].

2. General facts about vector fields on mapping spaces

2.1. **Tangent vectors and vector fields on a mapping space.** Consider the space of maps $\mathbf{Map}(M, N)$. Its "points" are smooth maps $\varphi\colon M \to N$. A *tangent vector* K at φ is an infinitesimal shift $\varphi \mapsto \varphi_\varepsilon = \varphi + \varepsilon K$, i.e.,

$$\varphi^i(x) \mapsto \varphi^i(x) + \varepsilon K^i(x) \quad (\varepsilon^2 = 0).$$

Hence K is a 'vector field along the map φ', i.e., a map $M \to TN$ that covers the map $\varphi\colon M \to N$ w.r.t. the projection $TN \to N$. As for ordinary manifolds, a tangent vector K to a mapping space $\mathbf{Map}(M, N)$ can be identified with its infinitesimal action on function(al)s δ_K, the variation along K; by definition,

$$S[\varphi + \varepsilon K] = S[\varphi] + \varepsilon\, \delta_K S \quad (\varepsilon^2 = 0).$$

Here δ_K is a linear operator mapping functionals to numbers. The familiar expansion

$$K = \delta_K = (-1)^{q\tilde{K}} \int_M Dx\, K^i(x) \frac{\delta}{\delta\varphi^i(x)},$$

may be regarded as the definition of variational derivatives $\frac{\delta}{\delta\varphi^i(x)}$. Here $\dim M = p|q$. Note, incidentally, that the parity of $\delta/\delta\varphi^i(x)$ is $\tilde{\imath} + q$ (not $\tilde{\imath}$). The role of the sign factor $(-1)^{q\tilde{K}}$ is in keeping the linearity of $\delta_K S$ in K w.r.t. the multiplication of K by odd scalars.

Hence, a *vector field* K on $\mathbf{Map}(M,N)$ gives infinitesimal shifts for arbitrary maps $\varphi \in \mathbf{Map}(M,N)$,

$$\varphi^i(x) \mapsto \varphi^i(x) + \varepsilon K^i[x|\,\varphi] \quad (\varepsilon^2 = 0)\,.$$

It is a functional on $\mathbf{Map}(M,N)$ taking values in tangent vectors, so that the value $K[\varphi]$ at φ is a tangent vector at φ. A vector field K can be identified with the corresponding variation δ_K, which is now an operator taking functionals to functionals:

$$K[\varphi] = (-1)^{q\tilde{K}} \int_M Dx\, K^i[x|\,\varphi]\, \frac{\delta}{\delta\varphi^i(x)}$$

and

$$\delta_K S\,[\varphi] = (-1)^{q\tilde{K}} \int_M Dx\, K^i[x|\,\varphi]\, \frac{\delta S[\varphi]}{\delta\varphi^i(x)}\,,$$

for a functional S. This is similar to writing vector fields on ordinary manifolds or supermanifolds as differential operators on functions.

The Lie bracket of vector fields on $\mathbf{Map}(M,N)$ is defined in the usual way. One either starts from the group commutator of the infinitesimal diffeomorphisms of $\mathbf{Map}(M,N)$, so that

$$1 + \varepsilon\eta\,[K_1,K_2] = (1+\eta K_2)^{-1}(1+\varepsilon K_1)^{-1}(1+\eta K_2)(1+\varepsilon K_1)\,,$$

or takes the (graded) commutator of the variations:

$$\delta_{[K_1,K_2]} = [\delta_{K_1},\delta_{K_2}] = \delta_{K_1}\delta_{K_2} - (-1)^{\tilde{K}_1\tilde{K}_2}\delta_{K_2}\delta_{K_1}\,.$$

In coordinates,

$$\begin{aligned}
[K_1,K_2] &= \int_M\int_M DxDy\left((-1)^{q\tilde{K}_2}K_1^j[y|\varphi]\,\frac{\delta K_2^i[x|\varphi]}{\delta\varphi^j(y)} - (-1)^{\tilde{K}_1\tilde{K}_2+q\tilde{K}_1}K_2^j[y|\varphi]\,\frac{\delta K_1^i[x|\varphi]}{\delta\varphi^j(y)}\right)\frac{\delta}{\delta\varphi^i(x)}\\
&= \int_M Dx\left((-1)^{q\tilde{K}_1}\delta_{K_1}K_2^i[x|\varphi] - (-1)^{\tilde{K}_1\tilde{K}_2+q\tilde{K}_2}\delta_{K_2}K_1^i[x|\varphi]\right)\frac{\delta}{\delta\varphi^i(x)}\,.
\end{aligned}$$

One can imagine various classes of vector fields on a mapping space corresponding to various types of the dependance of the components $K^i[x|\varphi]$ on φ. A particular case: $K^i[x|\varphi] = K^i(x,\partial\varphi(x),\partial^2\varphi(x),\ldots,\partial^s\varphi(x))$, i.e., the components $K^i[x|\varphi]$ are differential functions of φ at the same point x. Such vector fields K on $\mathbf{Map}(M,N)$ are known as *local vector fields*. (The 'evolutionary vector fields' of the jet-theoretic approach, cf. Olver [7], were made to mimic exactly this class of vector fields on mapping spaces.) Local vector fields are closed under commutator.

Remark 1. Everything above is completely standard at least in the case when M and N are ordinary manifolds. Our goal was mainly to recall the terminology and introduce the notation.

Remark 2. The concept of the mapping space $\mathbf{Map}(M,N)$ when M and N are supermanifolds and, in particular, its treatment as an infinite-dimensional supermanifold requires some comments. There are two aspects, the infinite dimensionality and being 'super', which are independent of each other. Even for ordinary manifolds, it has to be explained in which sense the set $\mathrm{Map}(M,N)$ of all smooth maps from a smooth manifold M to a smooth manifold N can be understood as an 'infinite-dimensional manifold'. We refer, for example, to the book [5] for one particular approach. In this paper, we follow the

'naive' or 'formal' viewpoint used by physicists and do not go into foundations. As for the supermanifold aspect, the subtlety is unrelated with the infinite-dimensionality. Note that in some cases, the mapping space $\mathbf{Map}(M,N)$ can be finite-dimensional. (For example, such is the mapping space $\mathbf{Map}(\mathbb{R}^{0|1},N)$ for any (super)manifold M, which coincides with the supermanifold ΠTN. Also, in general, $\mathbf{Map}(M,N)$ may contain finite-dimensional subspaces that are supermanifolds.) The key fact is that the mapping space $\mathbf{Map}(M,N)$ should be regarded as more than a set of smooth maps endowed with whatever structure.[3] Informally, 'odd parameters' should be allowed for these maps and, in the (general) case of infinite-dimensionality, these odd parameters are 'functional'. To avoid the discussion of an underlying topology and a structure sheaf for the space $\mathbf{Map}(M,N)$, the convenient formula

$$\mathrm{Map}(P,\mathbf{Map}(M,N)) = \mathrm{Map}(P\times M,N)$$

may be postulated as a working definition. Here P, M and N are supermanifolds, and for fixed M and N, and varying P, the r.h.s. serves as the definition of the l.h.s. as a functor of P. The set $\mathrm{Map}(P,\mathbf{Map}(M,N)) = \mathrm{Map}(P\times M,N)$ is, by definition, the set of all P-points of the supermanifold $\mathbf{Map}(M,N)$. It should be noted that whenever we refer to "points" of $\mathbf{Map}(M,N)$ or use set-theoretic notation, we always understand points in this generalized sense.

2.2. **Induced vector fields and the difference construction.** Diffeomorphisms of the source and target induce diffeomorphisms of the mapping space. For $F\in\mathrm{Diff}(M)$ and $G\in\mathrm{Diff}(N)$, we have the transformations F^* and G_* of $\mathbf{Map}(M,N)$,

$$F^*[\varphi]=\varphi\circ F\,,\quad G_*[\varphi]=G\circ\varphi\,,$$

which are the usual pull-back and push-forward of a map. Clearly,

$$(F_1\circ F_2)^* = F_2^*\circ F_1^*\,,\quad (G_1\circ G_2)_* = G_{1*}\circ G_{2*}\,.$$

The infinitesimal version of that holds for vector fields. For a map $\varphi\in\mathbf{Map}(M,N)$, vector fields on the source and target define its infinitesimal variations. Let $X\in\mathrm{Vect}(M)$ and let $Y\in\mathrm{Vect}(N)$. We can define vector fields X^* and Y_* on $\mathbf{Map}(M,N)$ by the formulas $X^*[\varphi]:=d\varphi\circ X$ and $Y_*[\varphi]:=Y\circ\varphi$, having in mind the pull-back and the push-forward of φ by the corresponding infinitesimal diffeomorphims:

$$\begin{aligned}\varphi+\varepsilon\,X^*[\varphi] &= (1_M+\varepsilon X)^*[\varphi] = \varphi\circ(1_M+\varepsilon X)\,,\\ \varphi+\varepsilon\,Y_*[\varphi] &= (1_N+\varepsilon Y)_*[\varphi] = (1_N+\varepsilon Y)\circ\varphi\,.\end{aligned}$$

From the definition follow the coordinate descriptions:

$$X^* = (-1)^{q\tilde{X}}\int_M Dx\,X^a(x)\,\partial_a\varphi^i(x)\,\frac{\delta}{\delta\varphi^i(x)}$$

and

$$Y_* = (-1)^{q\tilde{Y}}\int_M Dx\,Y^i(\varphi(x))\,\frac{\delta}{\delta\varphi^i(x)}\,.$$

In particular, both X^* and Y_* are local vector fields.

[3]We use boldface to distinguish $\mathbf{Map}(M,N)$ from such a set, which we denote $\mathrm{Map}(M,N)$. The set $\mathrm{Map}(M,N)$ with suitable topology is the underlying topological space for $\mathbf{Map}(M,N)$. If M and N are ordinary manifolds, there is no need for such a distinction between Map and **Map**.

Proposition 1. *For arbitrary* $X_1, X_2 \in \operatorname{Vect}(M)$*,*

$$[X_1, X_2]^* = -[X_1^*, X_2^*]\,.$$

For arbitrary $Y_1, Y_2 \in \operatorname{Vect}(N)$*,*

$$[Y_1, Y_2]_* = [Y_{1*}, Y_{2*}]\,.$$

For arbitrary $X \in \operatorname{Vect}(M)$ *and* $Y \in \operatorname{Vect}(N)$*,*

$$[X^*, Y_*] = 0\,.$$

The statement is obvious from the interpretation in terms of the infinitesimal diffeomorphisms. In particular, the commutativity of X^* and Y_* follows from the commutativity of the left and right shifts.

Corollary 1. *If* $Q_1 \in \operatorname{Vect}(M)$ *and* $Q_2 \in \operatorname{Vect}(N)$ *are homological vector fields, then the induced vector fields on* $\mathbf{Map}(M, N)$ *are also homological:*

$$(Q_1^*)^2 = 0\,, \quad (Q_{2*})^2 = 0\,.$$

For vector fields $X_1 \in \operatorname{Vect}(M)$ and $X_2 \in \operatorname{Vect}(N)$ of the same parity, define their *difference construction*, notation: $d(X_1, X_2)$, as the vector field on $\mathbf{Map}(M, N)$

$$d(X_1, X_2) := X_{2*} - X_1^*\,.$$

Or, equivalently,

$$d(X_1, X_2)[\varphi] := X_2 \circ \varphi - d\varphi \circ X_1\,.$$

The zeros of the vector field $d(X_1, X_2)$ are precisely such φ that X_1 and X_2 are φ-related.

Corollary 2. *If vector fields* $Q_1 \in \operatorname{Vect}(M)$ *and* $Q_2 \in \operatorname{Vect}(N)$ *are homological, then the vector field* $d(Q_1, Q_2)$ *on* $\mathbf{Map}(M, N)$ *is also homological.*

Indeed, Q_{2*} and Q_1^* commute and are homological. Therefore their difference is homological.

Remark 3. The notion of the difference construction (without such terminology) as a vector field on the space of maps and crucial Corollary 2 is due to [1].

Difference construction has nice properties. For example, suppose M_1, M_2 and M_3 are endowed with vector fields X_1, X_2 and X_3, resp. For any diagram

$$M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3\,,\,,$$

one may ask about the relation between the vector fields $d(X_1, X_3)$, $d(X_1, X_2)$ and $d(X_2, X_3)$.

Proposition 2. *The following identity holds:*

$$d(X_1, X_3)[\psi \circ \varphi] = d(X_2, X_3)[\psi] \circ \varphi + d\psi \circ d(X_1, X_2)[\varphi]\,. \tag{4}$$

Proof is straightforward.

Remark 4. For homological vector fields, the difference construction should be compared with the familiar definition of the differential on $\mathrm{Hom}(K, L)$ for (co)chain complexes K and L. Equation (4) should be compared with the Leibniz rule for this differential w.r.t. the composition of homomorphisms, to which it reduces in the case of complexes. Notions introduced in this section should be regarded as the non-linear analogs of the corresponding linear notions for complexes. (One can associate a Q-manifold to a cochain complex, so that the differential becomes a homological vector field, linear in coordinates.)

Equation (4) and similar identities can be used for introducing the important notion of a *Q-category*, generalizing the notion of a *Q-group* [10]. A *Q-category* is a smooth category such that the morphism 'sets' and possibly the 'set' of objects are Q-manifolds, maybe infinite-dimensional, and all the structure maps are Q-morphisms. As an example one can consider some category of Q-manifolds (morphisms — arbitrary smooth maps). It can be regarded as a *Q-category* w.r.t. the homological vector field $d(Q_\alpha, Q_\beta)$ for each (M_α, Q_α) and (M_β, Q_β). In particular, each supergroup of diffeomorphisms $\mathrm{Diff}\, M$ for a Q-manifold M is a *Q-group*.

As we already mentioned in the Introduction, there is an observation (AKSZ): the Hamiltonian vector field X_S corresponding to the AKSZ action is the difference construction for Q_1 and Q_2. (This explains $(S, S) = 0$.)

Question: is it possible, in general, to express the difference construction for arbitrary vector fields in a "Hamiltonian" or "gradient" form?

We shall deal with that in the next section.

3. Main construction and a proof

3.1. **Main statement.** Consider supermanifolds M and N and vector fields $X_1 \in \mathrm{Vect}(M)$ and $X_2 \in \mathrm{Vect}(N)$ of the same parity $\tilde{\varepsilon}$. Here we do not need them to be homological or anything. Therefore our analysis makes sense in the case of ordinary manifolds as well. In the previous section we introduced the difference construction $X_{12} = d(X_1, X_2)$ as a vector field on the mapping space $\mathbf{Map}(M, N)$. In coordinates,

$$X_{12} = (-1)^{q\tilde{\varepsilon}} \int_M Dx \left(X_2^i(\varphi(x)) - X_1^a(x) \frac{\partial \varphi^i}{\partial x^a} \right) \frac{\delta}{\delta \varphi^i(x)} \,. \tag{5}$$

We would like to ask a general question: is it possible to express this vector field on $\mathbf{Map}(M, N)$ in a "gradient form"? That means, it is possible to find a functional $S = S_{12}$ on $\mathbf{Map}(M, N)$ and coefficients $\Psi^{ij}(x, y)$ so that[4]

$$X_{12} = \pm \int_M Dx \, \Psi^{ij}(x, \varphi(x)) \, \frac{\delta S_{12}}{\delta \varphi^j(x)} \, \frac{\delta}{\delta \varphi^i(x)} \,? \tag{6}$$

Clearly, the answer depends on the particular X_1 and X_2, but we would like to find a "universal" construction that would work for "general" X_1 and X_2. The functional S_{12} should depend on X_1 and X_2, while the object $\Psi^{ij}(x, y)$, not. Of course this means the existence of a certain structure on the manifolds M and N. Our task is to identify this structure.

[4]More precisely, it is a "local gradient form" because the arguments in the integrand are taken at the same point $x \in M$. Note, incidentally, that we are bit sloppy with the common sign in (6) because it is not important here.

As we shall see, for this construction to hold, the vector fields X_1 and X_2, and the object $\Psi^{ij}(x,y)$ should obey certain constraints. We shall find them now.

First of all, let us realize what sort of geometric object $\Psi^{ij}(x,y)$ is. It lives on $M\times N$ and carries tensor indices from N. Recall that the variational derivative $\delta/\delta\varphi^i(x)$ transforms as a covector w.r.t. N and as a density of weight 1 (the component of a volume form) w.r.t. M. We shall apply the terminology such as "source density" and "target covector", and similar. Then $\Psi^{ij}(x,y)$ is a target tensor of rank 2 and a source density of weight -1 (so that to compensate the total weight of the two variational derivatives).

Note that no symmetry condition in the tensor indices is assumed a priori for $\Psi^{ij}(x,y)$.

Theorem 1. *In the setup described above, the difference construction $d(X_1,X_2)$ has a "universal gradient form"* (6) *if and only if the following holds. The matrix $\|\Psi^{ij}(x,y)\|$ is the inverse of $\|\omega_{ij}(x,y)\|$, which defines a target symplectic form and source density $\boldsymbol{\omega}$ of weight* 1 *on $M\times N$. The condition*

$$(L_{X_1}+L_{X_2})\boldsymbol{\omega}=0 \tag{7}$$

is satisfied and the action $S_{12}=S_{12}[\varphi,X_1,X_2]$ is given by the formula

$$S_{12}[\varphi,X_1,X_2]=\int_M \langle d\varphi\circ X_1, d_2^{-1}\boldsymbol{\omega}\rangle \;+\; d_2^{-1}\left(L_{X_1}d_2^{-1}\boldsymbol{\omega}+i_{X_2}\boldsymbol{\omega}\right) \tag{8}$$

(here d_1 and d_2 denote the exterior differentials on M and N, resp.).

The statement of the theorem deserves a few comments. (A sketch of a proof will follow in the next subsection.)

We can differentiate objects on $M\times N$ independently in the M- and N-directions, so it makes perfect sense to speak of a target symplectic form (so that $d_2\boldsymbol{\omega}=0$), as well as to apply to $\boldsymbol{\omega}$ the Lie derivatives w.r.t. vector fields on M and N. The Lie derivative L_{X_1} acts on $\boldsymbol{\omega}$ as on a volume form, while L_{X_2} acts on $\boldsymbol{\omega}$ as on a differential 2-form.

As it is clear from the presence of the inverses of the exterior differentials, equation (8) defines a functional which is multivalued. It will be seen from the proof that its variation is well-defined and each d^{-1} in the formula makes sense at least locally. We can check here that d_2^{-1} in the second term makes sense. Indeed, we need to check the d_2-closedness; we have $d_2\left(L_{X_1}d_2^{-1}\boldsymbol{\omega}+i_{X_2}\boldsymbol{\omega}\right)=L_{X_1}d_2d_2^{-1}\boldsymbol{\omega}+d_2i_{X_2}\boldsymbol{\omega}=L_{X_1}\boldsymbol{\omega}+L_{X_2}\boldsymbol{\omega}=0$, by (7).

Let us give a coordinate expression for the action S_{12}, which may be useful together with the coordinate-free formula (8). For simplicity we shall write the formulas in the purely even case. Everything extends without effort to the general super case. Suppose $\boldsymbol{\omega}=d_2\boldsymbol{\lambda}$ locally. Then $\boldsymbol{\lambda}$ is a source 1-form and target density. In local coordinates, $\boldsymbol{\lambda}=Dx\otimes dy^i\lambda_i(x,y)$ and $\omega_{ij}=\partial_i\lambda_j-\partial_j\lambda_i$. Then we have

$$S_{12}[\varphi,X_1,X_2]=\int_M Dx\,\left(X_1^a(x)\,\frac{\partial\varphi^i}{\partial x^a}\,\lambda_i\big(x,\varphi(x)\big)\;+U\big(x,\varphi(x)\big)\right), \tag{9}$$

where the "potential" $U(x,y)$ is a target scalar and source density. It is defined from the equation $\partial_jU(x,y)=\partial_a(\lambda_jX_1^a)+X_2^i\omega_{ij}$ (the integrability condition for it is exactly (7)).

Corollary. *If $\boldsymbol{\omega}=\boldsymbol{\rho}\,\omega$, where $\boldsymbol{\rho}=Dx\rho(x)$ is a volume element on M and ω is a symplectic form on N, then the condition $(L_{X_1}+L_{X_2})\boldsymbol{\omega}=0$ becomes $L_{X_1}\boldsymbol{\rho}=0$ and*

$L_{X_2}\omega = 0$ and we recover the AKSZ-type setup (for X_1, X_2 of arbitrary, but equal parity). The action S_{12} takes the form

$$S_{12}[\varphi, X_1, X_2] = \int_M \boldsymbol{\rho} \left(X_1^a \frac{\partial \varphi^i}{\partial x^a} \lambda_i(\varphi(x)) - H(\varphi(x)) \right).$$

Here $\lambda_i = \lambda_i(y)$, $d\lambda = \omega$, and $i_{X_2}\omega = -dH$.

3.2. **Sketch of a proof.** For the simplicity of notation consider the purely even case (extending to the general super case is straightforward). We are given that

$$X_{12} = \int_M \Psi^{ij}(x, \varphi(x)) \frac{\delta S_{12}}{\delta \varphi^j(x)} \frac{\delta}{\delta \varphi^i(x)}.$$

Since $X_{12}^i = X^i(y) - X^a(x) y_a^i$, the action $S = S_{12}$ can contain only first derivatives and should have the form

$$S = \int_M Dx \left(\Lambda_i^a(x, \varphi(x)) \frac{\partial \varphi^i}{\partial x^a} + U\big(x, \varphi(x)\big) \right),$$

so the Lagrangian is $L = \Lambda_i^a(x, y)\, y_a^i + U(x, y)$. Calculating the variational derivative $\delta S_{12}/\delta\varphi^j(x)$ we obtain

$$\frac{\partial L}{\partial y^j} - \frac{d}{dx^a}\left(\frac{\partial L}{\partial y_a^j}\right) = \frac{\partial \Lambda_k^a}{\partial y^j} y_a^k + \frac{\partial U}{\partial y^j} - \frac{d}{dx^a}\left(\Lambda_j^a\right) =$$
$$\frac{\partial \Lambda_k^a}{\partial y^j} y_a^k + \frac{\partial U}{\partial y^j} - \frac{\partial \Lambda_j^a}{\partial x^a} - \frac{\partial \Lambda_j^a}{\partial y^k} y_a^k =$$
$$\left(\partial_j \Lambda_k^a - \partial_k \Lambda_j^a\right) y_a^k + \frac{\partial U}{\partial y^j} - \frac{\partial \Lambda_j^a}{\partial x^a}.$$

We should have

$$\Psi^{ij}(x, y)\Big(\left(\partial_j \Lambda_k^a - \partial_k \Lambda_j^a\right) y_a^k + \partial_j U - \partial_a \Lambda_j^a\Big) =$$
$$- X^a(x) y_a^i + X^i(y) = -X^a(x) \delta_k^i y_a^k + X^i(y).$$

We arrive at the system

$$\Psi^{ij}\left(\partial_j \Lambda_k^a - \partial_k \Lambda_j^a\right) = -X^a \delta_k^i,$$
$$\Psi^{ij}\left(\partial_j U - \partial_a \Lambda_j^a\right) = X^i.$$

Note that $\Psi^{ij}(x, y)$ should be universal and not depend on X_1, X_2, while Λ_k^a and U should depend on X_1, X_2 by universal formulas. Therefore we conclude, from the first equation, that $\Psi^{ij} = \Psi^{ij}(x, y)$ is invertible. Introduce the inverse matrix ω_{ij}. We obtain

$$\partial_j \Lambda_k^a - \partial_k \Lambda_j^a = -\omega_{jk} X^a.$$

Therefore ω_{ij} is skew-symmetric. Likewise, we see that $\omega_{jk} = \partial_j \lambda_k - \partial_k \lambda_j$ for some λ_i. We conclude that $\Lambda_i^a = -\lambda_i X^a + \partial_i f^a$, where $f^a = f^a(x, y)$. Hence the Lagrangian is

$$L = \big(-\lambda_i(x, y) X^a(x) + \partial_i f^a(x, y)\big)\, y_a^i + U(x, y).$$

Note that $\partial_i f^a(x,y)\, y^i_a = D_a f^a - \partial_a f^a$, where D_a denotes total derivative w.r.t. x^a. Hence we can pass to an equivalent Lagrangian and re-define U by absorbing $-\partial_a f^a$:

$$L = -\lambda_i(x,y)X^a(x)\, y^i_a + U(x,y)\,.$$

We now look at the second equation from the system above. It gives, after contracting it with ω_{ki}, the equation

$$\partial_k U + \partial_a\big(\lambda_k X^a\big) = \omega_{ki} X^i$$

or

$$d_2 U + \partial_a\big(\boldsymbol{\lambda} X^a\big) = -i_{X_2}\boldsymbol{\omega} \;\Rightarrow\; \partial_a\big(\omega X^a\big) = -L_{X_2}\boldsymbol{\omega}\,.$$

Note, finally, that $\partial_a\big(\boldsymbol{\omega} X^a\big) = L_{X_1}\boldsymbol{\omega}$. Hence we arrive at the relation

$$L_{X_1}\boldsymbol{\omega} + L_{X_2}\boldsymbol{\omega} = 0$$

(this is a Lie derivative of an object on $M\times N$ w.r.t. the vector field $X_1 + X_2$). This is a necessary (and locally sufficient) condition for recovering U in the Lagrangian. For the action, we have arrived at the expression (9), up to an inessential common sign. This concludes the proof. □

4. Examples

Applications of the AKSZ construction are numerous. The following examples are recalled for illustration only.

Example 1 (see [1])**.** Consider a supermanifold M of dimension $n|m$. We can take ΠTM with the de Rham differential d as the homological vector field. This will be the source. The vector field d preserves the canonical volume form $D(x,dx)$ on ΠTM. For the target, consider a symplectic supermanifold N with the symplectic form ω of parity $n+m$. Suppose H is a function on N of parity $n+m+1$ satisfying $(H,H)=0$ w.r.t. the Poisson bracket generated by the symplectic structure. Let ω on N be locally $d\lambda$. Note that the parity of λ is $n+m+1$, which is the same as for H. On the mapping space $\mathbf{Map}(\Pi TM, N)$ we obtain the AKSZ action

$$S[\varphi,H] = \int_{\Pi TM} D(x,dx)\left(dx^a\frac{\partial\varphi^i}{\partial x^a}\,\lambda_i(\varphi(x,dx)) - H(\varphi(x,dx))\right)\,. \tag{10}$$

It satisfies $(S,S)=0$ w.r.t. the odd symplectic structure on $\mathbf{Map}(\Pi TM, N)$ given by the odd 2-form

$$\Omega[\varphi,\delta\varphi] = \int_{\Pi TM} D(x,dx)\;\omega(\varphi(x,dx),\delta\varphi(x,dx))\,.$$

The Hamiltonian vector field of S w.r.t. this structure is the difference construction for d and Q.

In the above example, maps $\Pi TM \to N$ can be interpreted as "N-valued forms" on M. Also, any such map φ naturally lifts to the map $\Pi TM \to \Pi TN$, which we denote by the same letter. This allows to consider pull-backs of forms on N to forms on M. Therefore, the functional given by (10) may be re-written simply as

$$S[\varphi,H] = \int_M \varphi^*(\lambda - H) \tag{11}$$

(in the r.h.s. we have the integral of a pseudodifferential form over M).

The following example is a particular case of the previous one.

Example 2 (Cattaneo and Felder [2, 3])**.** (It is convenient to change notation slightly.) Consider the space of maps $\Pi TD \to \Pi T^*M$, where $D = D^2$ is a 2-disk with boundary, and M is a Poisson manifold with an even bracket. It is specified by a function P on ΠT^*M (the Poisson bivector), which is fiberwise quadratic and satisfies $(P, P) = 0$ w.r.t. the canonical odd Poisson bracket on ΠT^*M given by the canonical odd symplectic form $\omega = -dx^a dx^*_a = d(dx^a\, x^*_a)$. The de Rham differential as a vector field on ΠTD preserves the canonical volume form $\boldsymbol{\rho} = D(u, du)$, where u^i are coordinates on the disk. The homological vector field X_P on ΠT^*M which is the odd Hamiltonian field corresponding to the even function $P = \frac{1}{2}\, P^{ab} x^*_b x^*_a$ is nothing but the Lichnerowicz differential for the Poisson cohomology of (M, P). The AKSZ action written as in (11) is

$$S[\varphi, P] = \int_{D^2} \varphi^*\!\left(dx^a x^*_a - \frac{1}{2}\, P^{ab}(x) x^*_b x^*_a\right) . \tag{12}$$

It turns out that the mapping space $\mathbf{Map}(\Pi TD, \Pi T^*M)$ plays the role of the extended phase space of the Batalin–Vilkovisky method. Namely, one starts from the space of the vector bundle maps $\mathbf{Hom}(\Pi TD, \Pi T^*M) \subset \mathbf{Map}(\Pi TD, \Pi T^*M)$. It turns out that here

$$\mathbf{Map}(\Pi TD, \Pi T^*M) \cong T^*\left(\mathbf{Hom}(\Pi TD, \Pi T^*M)\right)$$

and the AKSZ action (12) on the full space $\mathbf{Map}(\Pi TD, \Pi T^*M)$ plays the role of the Batalin–Vilkovisky extended action w.r.t. to the same action restricted to the subspace of vector bundle morphisms $\mathbf{Hom}(\Pi TD, \Pi T^*M)$. Using this method, Cattaneo and Felder showed how to obtain Kontsevich's formulas for deformation quantization of a Poisson manifold (M, P).

As mentioned, the above examples are in no way new (maybe the exposition is slightly non-standard). It would be interesting to obtain an example where, as in the previous section, there is no given factorization $\boldsymbol{\omega} = \boldsymbol{\rho} \otimes \omega$ and objects naturally live on $M \times N$. This may happen if one replaces maps $M \to N$ by sections of of a fiber bundle and $M \times N$ by the total space.

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